Premium Auctions and Risk Preferences[1]

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By Audrey Hu, Theo Offerman, and Liang Zou

In a premium auction, the seller offers some "pay back," called premium, to the highest bidders. This paper investigates how the performance of such premium tactic is related to the participants' risk preferences. By developing an English premium auction model with symmetric interdependent values, where both the seller and the buyers may be risk averse (or preferring), we show that a) the premium reduces the riskiness of revenue regardless of the bidders' risk preferences, and b) the premium causes the expected revenue to increase in the bidders' risk tolerance. A "net-premium effect" and a "second-order stochastic dominance effect" are key to these results.

Keywords: premium auction, English auction, risk preferences, premium effects

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1 Introduction

Premium auctions are featured by the seller committing to some pre-specified rule that rewards certain high bidders. In Europe, such practice dates back to the Middle Ages, and it continues to be employed nowadays in the sales of houses, land, boats, machinery, airplanes or inventory of insolvent businesses from time to time (e.g., Goeree and Offerman, 2004).

The use of this type of auctions may be puzzling because in the canonical symmetric independent private values model with risk neutral buyers the premium is irrelevant: it would yield the same expected revenue as any other efficient auction (e.g., Myerson, 1981). The "Santa Claus Auction" considered in Riley and Samuelson (1981) is a fine example of premium auctions that illustrates this revenue equivalence principle. In order to understand the purpose of a premium, existing studies have identified situations where such tactic may enhance expected revenue, that is, where the bidders exhibit strong asymmetries prior to the auction. Notably, these studies have limited attention to the special case where one strong bidder (or cartel) competes with several weak bidders (e.g., Goeree and Offerman, 2004; Milgrom, 2004; Hu, Offerman and Onderstal, 2010). In such a situation, the premium serves to lure the weak bidders–who otherwise would have no real interest in participating in the auction–to show up and bid up the price that may result in a higher net profit to the seller. Other tactics of a similar sort, such as using "bidding credits," "set-asides," or "subsidies" have also been shown to enhance competition when bidders are asymmetric (e.g., Ayres and Cramton, 1996; Rothkopf et al., 2003; Athey and Levin, 2006).[1] Beyond the specific assumption made in these models, "the case for premium auctions remain uncertain (Milgrom, 2004, p.241)."

In this paper, we develop a model of English premium auctions (henceforth, EPA) in the classic setting of symmetric and interdependent values (Milgrom and

[1] These results are consistent with Myerson 1981, whose classic work shows that expected revenue maximization leads to policies that favor disadvantaged bidders.

Weber, 1982), allowing both the seller and the bidders to be risk averse (or preferring).[2] [3] The EPA we consider proceeds in two stages. In the first stage, the seller raises the price until all but two bidders (finalists) have withdrawn. In the second stage, the price is raised further until one finalist withdraws. The remaining finalist wins the object and pays the price at which the auction concludes. In addition, both finalists receive a premium determined by a pre-specified function of the difference between the ending prices of the two stages.[4]

Like English auctions (henceforth, EA), an appealing feature of the EPA is that it is detail-free to the seller as long as the premium rule is exogenously given. Apart from that, when a reserve price is desirable but the seller lacks sufficient knowledge about the buyers' value distribution, or the ability to commit to a reserve price, the first-stage ending price in an EPA can serve as a *de facto* reserve price for the seller. The role of the first stage of an EPA is therefore to efficiently elicit a reserve price through competition of the bidders—similar to that of the two-stage Anglo-Dutch auction proposed by Klemperer (2002).[5]

A noteworthy result in this paper is what we call the "net-premium effect"

[2]In what follows, risk preferring (or risk loving, risk seeking) refers to the case where the utility function is convex. For instance, a bidder can be an agent of a buyer, whose marginal commission by contract increases in the buyer's payoff.

[3]Our model is thus related to models of standard auctions with risk averse bidders, e.g., Holt (1980), Riley and Samuelson (1981), Harris and Raviv (1981), Milgrom and Weber (1982), Matthews (1983, 1987), Maskin and Riley (1984), Cox et al. (1982, 1988), Smith and Levin (1996), and Eso and White (2004); and to models with a risk averse seller, e.g., Waehrer et al. (1998) and Eso and Futo (1999). See also Hu, Matthews and Zou (2010) for an analysis of standard auctions with independent private values where both the seller and the buyers are risk averse.

[4]Our model is a generalization of the "Amsterdam Second-Price Auction" developed in Goeree and Offerman (2004) for the case with independent private values, linear premium rules, and risk neutral bidders.

[5]Because of the premium, however, the reserve price elicited from the first stage of an EPA will be higher than that from an EA.

(Theorem 4). The theorem shows that even for nonlinear utility functions, as long as the bidders have independent signals the premium's influence on each bidder's expected utility can be separated from that of a standard EA as a consequence of incentive compatibility. When separately calculated, the net-premium effect predicts that every bidder's expected utility for the premium, conditional on all updated information, equals his utility level if he quits at the current price. This finding is interesting on its own, as it adds a new insight into the revenue equivalence theorem, i.e., if the bidders are risk neutral, then revenue equivalence implies that the ex ante *expected premium* a bidder might receive is zero. Now with risk averse (or preferring) bidders, we have a more general conclusion that the ex ante *expected utility for the premium* is zero. Akin to the revenue equivalence theorem in the risk neutral context, our net-premium effect theorem offers as well a useful tool for the comparative welfare analysis of the premium auctions beyond risk neutrality.

Starting with the second-stage, we characterize the EPA equilibrium and derive the basic properties of the bid function in such an equilibrium (Theorem 1). Next we prove that the bid function exists and, moreover, has to be unique (Theorem 2). The existence of the EPA symmetric equilibrium is then established in Theorem 3. This leads to our Proposition 1, i.e., for any arbitrary premium function, the expected revenue increases in bidders' risk tolerance. Since the expected revenue will be invariant with the premium when the bidders are risk neutral, Proposition 1 implies that a risk neutral seller is better (worse) off to offer a premium if the bidders are risk preferring (averse). In other words, for a risk neutral seller facing risk averse buyers, expected revenue maximization cannot be the reason for choosing a premium auction.

Another noteworthy result is identified as the "second-order stochastic dominance effect" or "SSD effect" (Theorem 5). This effect says that the premium, in general, reduces the riskiness of revenue and the bidders' payoffs in the sense of second-order stochastic dominance (SSD). Therefore, for a risk-averse seller facing risk-neutral buyers (e.g., the cases considered in Waehrer et al. 1998), the EPA is more attractive than the EA (Proposition 2).

From the bidders' perspective (cf. Matthews, 1987; Smith and Levin, 1996), we show in Proposition 3 that under certain plausible conditions, the risk averse (preferring) bidders have higher (lower) expected utilities in an EPA than an EA. Hence, the conventional wisdom that premium auctions tend to attract risk-seeking speculators does not apply in our context.

In general, our results show how the performance of a premium auction varies with the set of the seller and the buyers' risk preferences—in some cases outperforms, and in other cases underperforms, the standard auctions. In this respect, it is interesting to note that some auction houses do switch repeatedly between using the premium auctions and the standard Ebay-like auction procedures.[6]

The rest of the paper is organized as follows. Section 2 presents the EPA model and some preliminary results. Section 3 characterizes, and establishes the existence and uniqueness, of an EPA symmetric equilibrium. A closed-form equilibrium solution is derived in this section for the case where the bidders have private values and exhibit constant absolute risk aversion (CARA). Section 4 derives the net-premium effect, the SSD effect, and analyzes their consequences for the expected revenue and the players' expected utilities. Section 5 concludes. Some proofs are relegated to the Appendix to improve readability.

2 The Model

A single object is to be sold to one of n (> 2) bidders via an English premium auction (EPA)[7] in a generalized symmetric setting à la Milgrom and Weber (1982). Each

[6]For instance, Troostwijk adopted a premium auction for its recent sale of a Boeing 737-400 in November 2009, although the auction house stays with the standard procedures more often. See http://www.troostwijkauctions.com/nl/ for more examples.

[7]We focus on the English-type of premium auctions in this study mainly because of their popularity in practice.

bidder $i \in N = \{1, 2, ..., n\}$ receives a private signal $s_i \in [L, H] \subset \mathbb{R}$ prior to the auction. Given s_i, the signals of the bidders other than i will be denoted by s_{-i}. The monetary payoff of the auctioned object to buyer i is $v^i = v(s_i, s_{-i})$. As in Milgrom and Weber (1982),[8] we assume that v is invariant to permutations of its last $n-1$ arguments (i.e., for all permutations $\hat{s}_{-i}$ of s_{-i}, $v(s_i, s_{-i}) = v(s_i, \hat{s}_{-i})$). The function $v : [L, H]^n \to \mathbb{R}$ is assumed to be twice continuously differentiable, with $v_1 > 0$ and $v_j \geq 0$ for $j \in \{2, ..., n\}$.[9] Ex ante, the signals $s_1, ..., s_n$ are independently distributed on $[L, H]$ according to the same distribution function F, which has a density function $f = F'$ that is strictly positive on $[L, H]$ and continuously differentiable on (L, H).[10] Given any vector of signals s, let $s_{(1)}, s_{(2)}, ..., s_{(n)}$ denote the lowest, second lowest,..., highest signals from among s.[11]

Although the auction can be conducted incessantly until the object is sold, it is equivalent and analytically convenient to perceive it, as we do, as a two-stage auction. In the first stage, a price for the object rises continuously from a sufficiently low level and each bidder stays in the auction until he chooses to quit (e.g., by pressing an electronic button). This stage rounds up as soon as only two bidders, called *finalists*, remain and the price level (X) at which the last bidder quits, called the *bottom price*, will serve as a reserve price onwards. In the second stage, the price level rises from X until one of the finalists quits. The last one who stays wins

[8]See also Levin and Smith (1996), Waehrer et al. (1998), and Eso and White (2004) for related models.

[9]For functions with two or more variables, we use subscripts to denote their partial derivatives with respect to the corresponding variable.

[10]It is well-known that the revenue equivalence theorem depends on the assumption that the players' signals are independently distributed (e.g., Myerson 1981; Riley and Samuelson, 1981; Waehrer et al., 1998; and Krishna and Maenner, 2001). For the same reason we focus on independent signals in order to obtain the "net-premium effect" (Theorem 4) that is essential for the subsequent predictions in this study.

[11]We denote by $s_{(1)}$ the lowest order statistic (rather than the highest) for the pnemonic hint that the bidder with the lowest signal will drop out first at a price p_1, etc.

the object and pays the price (b) at which the other finalist quits. In addition, both finalists receive a cash premium from the seller that is equal to $\varphi(b-X)$,[12] where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a twice continuously differentiable function such that $\varphi(0) = 0$, $0 < \varphi' \leq 1/2$, and $\varphi'' \leq 0$. We call such φ a *premium function.* As usual, ties are assumed to be resolved randomly in both stages. If two or more bidders simultaneously withdraw at price X in the first stage, with only one bidder left, then a random device will choose one of these bidders to be a finalist. If both finalists withdraw at the same price b, then both will receive a premium equal to $\varphi(b-X)$, and one of them will be randomly chosen to receive the object and pays the price b. Clearly, if $\varphi \equiv 0$ then the model reduces to a standard English auction—with a possible (coffee) break when only two bidders remain.

The bidders have the same utility function, $u : \mathbb{R} \to \mathbb{R}$, that is strictly increasing and twice continuously differentiable. We further assume that u is log-concave in that for all $z \in \mathbb{R}$, $\ln[u(w)-u(z)]$ is weakly concave in w on (z,∞). This condition is commonly invoked for the existence of equilibria in the first-price sealed-bid auctions (e.g., Athey (2001)), which allows the bidders to be risk neutral, risk averse, and to some extent risk preferring. If a bidder quits before entering the second stage, he pays and receives nothing, in which case his utility is normalized to be $u(0) = 0$. We also normalize the seller's reservation value for the object to be zero, and assume that $v(L, ..., L) \geq 0$ so that the sale will take place with certainty.

For some of the results we will use the following lemma. It is a variation of the "Ranking Lemma" of Milgrom (2004; p.124). See also Milgrom and Weber (1982, Lemma 2) and Hu, Matthews and Zou (2010, Lemma 1).

Lemma 1 *For* $-\infty < c < d < \infty$ *and* $h : [c,d] \to \mathbb{R}$ *continuous with* $h(d) \geq 0$,

[12] A virtually equivalent model is to award the premium only to the highest losing bidder. This will lead to a different equilibrium bid function than the one derived in Theorem 1, but the qualitative conclusions of the subsequent theorems and propositions will remain the same. See footnote 15.

$$(i)\ \textit{if} \begin{cases} h \text{ is differentiable on } [c,d] \text{ and} \\ [\forall t \in [c,d], h(t) = 0 \Rightarrow h'(t) < 0] \end{cases}, \text{ then } h > 0 \text{ on } [c,d);$$

$$(ii)\ \textit{if} \begin{cases} h \text{ is differentiable on } [c,d) \text{ and} \\ [\forall t \in [c,d),\ h(t) \leq 0 \Rightarrow h'(t) \leq 0] \end{cases}, \text{ then } h \geq 0 \text{ on } [c,d].$$

Proof. The proof of part (i) is analogous to Hu, Matthews and Zou (2010; Lemma 1(i)), hence is omitted. To show part (ii), assume that h is differentiable on $[c, d)$. Suppose $h(t) < 0$ for some $t \in [c, d)$. Then the continuity of h and the assumption $h(d) \geq 0$ imply the existence of $\hat{t} \in (t, d]$ such that $h(\hat{t}) = 0$ and $h(s) < 0$ for all $s \in [t, \hat{t})$. By the mean value theorem, this implies that there exists $s \in (t, \hat{t})$ such that $h'(s) = \left[h(\hat{t}) - h(t)\right] / (\hat{t} - t) > 0$. Therefore, the hypothesis in the square brackets of part (ii) does not hold. ■

3 The EPA Equilibrium

3.1 Bidding strategies

Suppose the first stage is over and a price history $p_1 \leq ... \leq p_{n-2}$ has been publicly observed, where the price p_1 indicates the level at which the first bidder dropped out, ..., and p_{n-2} the level at which the last bidder dropped out in the first stage. Suppose that the low-signal bidders drop out first, and that the price at which the bidder drops out allows all active bidders to infer his true signal. Then, a finalist's second-stage strategy can be described by a *bid function* of his signal, conditional on the bottom price $X = p_{n-2}$ and the revealed signals $\bar{s} = (s_{(1)}, ..., s_{(n-2)})$,

$$b(\cdot|\bar{s}; X) : [\max \bar{s}, H] \to [X, \infty) \tag{1}$$

For a finalist with signal x, $b(x|\bar{s}; X)$ specifies the price at which he will quit. We say that b is a *second-stage EPA equilibrium* if conditional on $(\bar{s}, X)$, adopting $b(\cdot|\bar{s}, X)$ maximizes each finalist's expected utility given that the other finalist adopts the same strategy b. Theorem 2 will show that such an equilibrium exists, and is unique given any updated information $(\bar{s}, X)$. Uniqueness is important because it ensures that the bidders in the first stage will have the same expectation about the second-stage equilibrium.

Because the active bidders in the first stage compete to enter the second stage, the anticipated second-stage equilibrium will affect their first-stage strategies. Theorems 1 and 3 will show that the equilibrium bid function $b(x|\bar{s}, X)$ necessarily has the property of being continuous, strictly increasing in x, weakly increasing in each component of $\bar{s}$, and that $b(x|\bar{s}; X) - X$ is strictly decreasing in X as long as $b(x|\bar{s}; X) > X$. These properties allow us to define the first-stage strategy $\left(b_{(1)}, ..., b_{(n-2)}\right)$ induced from b iteratively by

$$
\begin{gathered}
b_{(1)}(x; p) = b(x|x, ..., x; p) \quad (b_{(1)}(x; p_1) = p_1 \Rightarrow x = s_{(1)}) \\
b_{(2)}(x; p) = b(x|s_{(1)}, x, ...x; p) \quad (b_{(2)}(x; p_2) = p_2 \Rightarrow x = s_{(2)}) \\
\cdots \\
b_{(k)}(x; p) = b(x|s_{(1)}, ..., s_{(k-1)}, x; p) \quad (b_{(k)}(x; p_k) = p_k \Rightarrow x = s_{(k)})
\end{gathered}
\tag{2}
$$

for $k \leq n-2$, where p denotes the ongoing price.

The first-stage strategy can be described as follows. If a bidder has signal y and adopts strategy $\left(b_{(1)}, ..., b_{(n-2)}\right)$, then he will drop out at a price p whenever $b_{(1)}(y, p) = p$, if no other bidders have dropped out yet. The property of $b_{(1)}$ will allow the other bidders to infer that $y = s_{(1)}$ from the price at which the bidder quits. If the bidder remains active where $k-1$ bidders have dropped out, then conditional on the revealed signals $s_{(1)}, ..., s_{(k-1)}$ the bidder will drop out at a price $b_{(k)}(y, p) = p$, and so on.

We say that $\bar{b} = (b_{(1)}, ..., b_{(n-2)}, b)$ is an *EPA symmetric equilibrium* (or EPA equilibrium for short) if (i) $b(\cdot|\bar{s}; X)$ is a second-stage equilibrium conditional on any signal vector $\bar{s}$ and bottom price X; and (ii) in the first stage, conditional on any updated information, it is optimal for each bidder to adopt strategy $\bar{b}$ providing the other bidders adopt $\bar{b}$.[13]

3.2 The second-stage equilibrium

We analyze the EPA equilibrium strategies by backward induction, starting with the second-stage with updated information $\bar{s} = (s_{(1)}, ..., s_{(n-2)})$ and $X = p_{n-2}$.

Consider a second-stage EPA that is going on at price $p \geq X$. The (yet to be verified) property of $b(\cdot|\bar{s}, X)$ implies that there is an r solving $b(r|\bar{s}, X) = p$ such that if both finalists remain in the auction they must have signals higher than r. For $x \geq r$, $[F(x) - F(r)] / [1 - F(r)]$ is thus each finalist's updated probability that the other finalist has a lower signal than x. We call such r the *current screening level* (or screening level for short), which is implicitly defined through its one-to-one relation with the ongoing price p. This allows us to directly refer to the screening level r rather than the price level p in describing an ongoing second-stage EPA.

If both finalists adopt the bid function $b(\cdot|\bar{s}, X)$, then each with signal $x \geq r$ will have a conditional expected utility equal to

$$\begin{aligned}
&U(x|r, \bar{s}, X) \\
&\equiv \frac{1}{1 - F(r)} \int_r^x u\left(v(x, y, \bar{s}) - b(y|\bar{s}, X) + \varphi(b(y|\bar{s}, X) - X)\right) dF(y) \\
&+ \frac{1 - F(x)}{1 - F(r)} u\left(\varphi(b(x|\bar{s}, X) - X)\right),
\end{aligned} \tag{3}$$

where, on the right-hand side of (3), the first term comes from the event that the bidder wins and the second term from the event that the bidder loses.

An interesting aspect of the premium auction is that the premium in effect introduces an additional "affiliation" of the payoffs to the two finalists. A higher signal of the opponent can now be "good news" for two reasons: increasing the expected premium and increasing the value for each bidder. There are also some important

[13] We focus on symmetric equilibria in this study. It is known that even in ex ante symmetric settings there may exist multiple asymmetric equilibria (e.g., Maskin and Riley, 2003).

(strategic) differences between EPA and the standard EA with interdependent values (e.g., Milgrom and Weber, 1982; Eso and White, 2004). For instance, when a bidder decides to drop out at a price in an EA "he would be just indifferent between winning [at a tie] and losing at that price." (Milgrom and Weber, 1982, p.1105). In an EPA, however, should both finalists simultaneously drop out they would both prefer losing rather than winning. This is due to the fact that the equilibrium bid in the EPA is strictly higher than a bidder's value if he wins at a tie (Theorem 1).

Another fact highlighting the difference between EA and EPA is that the EA equilibrium strategy calls for each bidder to stay in the auction until his expected utility conditional on winning in a tie is zero (or equal to his status-quo utility that is commonly known beforehand). This property implies a straightforward solution for the symmetric bid function of the EA. However, in the second-stage EPA, a finalist will drop out at a price level at which his utility is positive due to the premium collected. But unlike in the EA, this utility level at which the bidder quits is private information in the EPA and cannot be determined beforehand unless the bid function is already given. This fact illustrates a difficulty of applying the standard EA analysis to the EPA.

Instead, we appeal to an argument of Milgrom and Weber (1982, p. 1105), which implies, in our setting, that despite the interdependency of the utility payoffs the second-stage EPA can be seen as a strategically equivalent second-price sealed bid (or Vickrey) premium auction. This result derives from the fact that when there are only two bidders remain, the bid function cannot be made dependent on the revealed signal of the opponent because it will be too late for the winner to adjust his bid when the opponent drops out. By the same logic, this argument applies to any price level p of the second-stage EPA as long as the auction continues. That is, the auctioneer can switch from an EPA at any price $p \geq X$ to a Vickrey premium auction with reserve price p without affecting the expected outcome. The Vickrey premium auction in our setting will stipulate that the bidders submit sealed bids no less than the current price p, and that the highest bidder wins the object and pays the bid b of the opponent while both finalists receive a premium $\varphi(b-X)$. We shall follow this approach from now on in the analysis of the second-stage EPA.

As long as the auction continues, given the updated screening level $r \in [s_{(n-2)}, H)$, the expected utility of a finalist who has signal $x \in [r, H]$ and who bids as though his signal is $z \in [r, H]$ equals

$$\begin{aligned}
&\overline{U}(x, z|r, \bar{s}, X) \\
&\equiv \frac{1}{1-F(r)} \int_r^z u\left(v(x, y, \bar{s}) - b(y|\bar{s}, X) + \varphi(b(y|\bar{s}, X) - X)\right) dF(y) \\
&+ \frac{1-F(z)}{1-F(r)} u\left(\varphi(b(z|\bar{s}, X) - X)\right), . \qquad (4)
\end{aligned}$$

Hence, given the updated information $(r, \bar{s}, X)$, equilibrium (or incentive compatibility) holds in the second stage if and only if for all $r \in [s_{(n-2)}, H)$, and $x, z \in [r, H]$,

$$U(x|r, \bar{s}, X) \geq \overline{U}(x, z|r, \bar{s}, X). \qquad (5)$$

Our first theorem provides the characterization of an EPA second-stage equilibrium. The proof of this theorem is lengthy, hence is relegated to the Appendix.

Theorem 1 (Necessary and sufficient condition) *Given any signal vector $\bar{s} \in [L, H]^{n-2}$ and bottom price $X \leq v(H, H, \bar{s})$, the bid function $b(\cdot|\bar{s}, X) : [\max \bar{s}, H] \to [X, v(H, H, \bar{s})]$ is a second-stage EPA equilibrium if and only if b is the solution of the following differential equation (6), satisfying $b_1 > 0$ and the boundary condition (7):*

$$b_1(x|\bar{s}, X) = B(b, x|\bar{s}, X) \frac{f(x)}{1-F(x)}, \quad x \in [\max \bar{s}, H) \qquad (6)$$

$$b(H|\bar{s}, X) = \lim_{x \uparrow H} b(x|\bar{s}, X) = v(H, H, \bar{s}) \qquad (7)$$

where

$$B(b, x|\bar{s}, X) = \frac{u\left(\varphi(b-X)\right) - u\left(v(x, x, \bar{s}) - b + \varphi(b-X)\right)}{u'\left(\varphi(b-X)\right)\varphi'(b-X)} \qquad (8)$$

Remark 1: Note that the derived bid function b is independent of any screening level r. This suggests that updating the screening level as the price increases will not affect each finalist's bidding behavior. Indeed, this observation lies at the bottom of the argument that the English (premium) auction and the Vickrey (premium) auction are strategically equivalent when there are only two bidders remain.

Remark 2: It is easy to see from (6) and (8) that $b_1 > 0$ is equivalent to $b(x|\bar{s}, X) > v(x, x, \bar{s})$. This means that regardless of their risk preferences the premium induces the bidders to bid higher then their values upon winning in a tie.

Remark 3: Because the right-hand side of (6) is continuously differentiable in b, X, and each component of $\bar{s}$, the solution $b(x|\bar{s}, X)$ is continuously differentiable in these variables on their relevant definition domain (e.g., Hale, 2009; Chapter 1, Theorem 3.3).

Remark 4: Because the right-hand side of (6) is continuously differentiable in b and x (except at $x = H$), by the mean value theorem we can write, for $x \in (\max \bar{s}, H)$,

$$b_1(x|\bar{s}, X) = \frac{u'(\xi)\,[b - v(x, x, \bar{s})]}{u'(\varphi(b - X))\,\varphi'(b - X)} \frac{f(x)}{1 - F(x)},$$

where $\xi \to \varphi(v(H, H, \bar{s}) - X)$ as $x \uparrow H$ (hence $b \to v(H, H, \bar{s})$). For convenience, define $v'(x, x, \bar{s}) = v_1(x, x, \bar{s}) + v_2(x, x, \bar{s})$. Since the above expression has a 0/0 form at $x = H$, by L'Hospital's rule and taking limit as $x \uparrow H$ yields

$$\begin{aligned}\lim_{x \uparrow H} b_1(x|\bar{s}, X) &= \frac{f(H)}{\varphi'(v(H, H, \bar{s}) - X)} \lim_{x \uparrow H} \frac{b(x|\bar{s}, X) - v(x, x, \bar{s})}{1 - F(x)} \\ &= -\frac{\lim_{x \uparrow H} b_1(x|\bar{s}, X) - v'(H, H, \bar{s})}{\varphi'(v(H, H, \bar{s}) - X)}.\end{aligned}$$

This allows us to denote by $b_1(H|\bar{s}, X)$ the value of the continuous extension of $b_1(x|\bar{s}, X)$ at $x = H$:

$$b_1(H|\bar{s}, X) = \lim_{x \uparrow H} b_1(x|\bar{s}, X) = \frac{v'(H, H, \bar{s})}{1 + \varphi'(v(H, H, \bar{s}) - X)}. \tag{9}$$

This property will be useful later on.

Our next theorem concerns the existence and uniqueness of an equilibrium bid function b. Because the right-hand side of differential equation (6) is undefined at $x = H$ (let alone Lipschitzian at this boundary point), we cannot directly apply the fundamental theorem of ordinary differential equations for a (unique) solution. The existence of a solution of (6)-(7) can be readily established, however, by employing standard techniques from ordinary differential equations theory. On the other hand, we can also look for sufficient conditions that directly guarantee the existence of an EPA equilibrium. As shown in Athey (2001), the set of sufficient conditions for the existence of equilibria in a large class of games of incomplete information includes a single crossing condition as proposed by Milgrom and Shannon (1994). In our context, the Milgrom-Shannon single crossing condition holds as long as $\overline{U}(x, z|r, \bar{s}, X)$ is supermodular in (x, z).[14] We take this (shorter) approach in Theorem 2, verify that $\overline{U}(x, z|r, \bar{s}, X)$ is supermodular, and further show that the EPA equilibrium is unique.

Theorem 2 (Existence and uniqueness) *Given any signal vector* $\bar{s} \in [L, H]^{n-2}$ *and bottom price* $X \leq v(H, H, \bar{s})$, *there exists a unique second-stage equilibrium* $b(\cdot|\bar{s}, X) : [\max \bar{s}, H] \to [X, v(H, H, \bar{s})]$.

Proof. (**Existence**) By Theorem 1, there is no loss of generality to restrict attention to differentiable and strictly increasing bid functions in search of an equilibrium. The case with $X = v(H, H, \bar{s})$ is trivial, since it is then common knowledge that both finalists have values equal to H and they will bid $v(H, H, \bar{s})$. Now suppose $X < v(H, H, \bar{s}))$ and that the opponent of a finalist adopts an increasing and differentiable bid function b. Then this finalist's expected utility at any screening level $r \geq \max \bar{s}$ is given by $\overline{U}(x, z|r, \bar{s}, X)$, where x is the bidder's true signal and z determines his bid $b(z|\bar{s}, X)$. Since b is continuous and strictly increasing in z, without ambiguity

[14]See Milgrom and Weber (1982) and Athey (2001) for this result and for the definition of supermodularity.

we can treat z as the bidder's "action." It follows from (4) that

$$\begin{aligned}&\overline{U}_{12}\left(x, z|r, \bar{s}, X\right)\\&=\frac{f(z)}{1-F(r)}u'\left(v(x,z,\bar{s})-b(z|\bar{s},X)+\varphi(b(z|\bar{s},X)-X)\right)v_1(x,z,\bar{s})>0.\end{aligned}$$

This inequality implies that $\overline{U}$ is supermodular in (x, z) (Topkis (1978)). Thus, by Athey (2001; Corollary 2.1), there exists an increasing second-stage equilibrium b of EPA. (It is easy to verify that the other assumptions of Athey's Corollary 2.1 are satisfied in our context.)

(**Uniqueness**) The right-hand side of (6) strictly increases in b as can be seen from

$$\begin{aligned}&B_1(b, x|\bar{s}, X)\\&=1+\frac{u'\left(v-b+\varphi\right)}{u'\left(\varphi\right)}\frac{1-\varphi'}{\varphi'}-\frac{u\left(\varphi\right)-u\left(v-b+\varphi\right)}{u'\left(\varphi\right)\varphi'}\left(\frac{u''\left(\varphi\right)\varphi'}{u'\left(\varphi\right)}+\frac{\varphi''}{\varphi'}\right)\\&\geq\frac{u'\left(v-b+\varphi\right)}{u'\left(\varphi\right)}\frac{1-\varphi'}{\varphi'}+\left(1-\frac{u\left(\varphi\right)-u\left(v-b+\varphi\right)}{u'\left(\varphi\right)}\frac{u''\left(\varphi\right)}{u'\left(\varphi\right)}\right) &(10)\\&\geq\frac{u'\left(v-b+\varphi\right)}{u'\left(\varphi\right)}\frac{1-\varphi'}{\varphi'} &(11)\\&>0 &(12)\end{aligned}$$

where (10) is due to $\varphi'' \leq 0$, and (11) derives from u being log-concave, using the fact that $v \leq b$ (so that the term in the brackets in (10) is nonnegative).

Now suppose for some $\bar{s} \in [L, H]^{n-2}$ and $X < v(H, H, \bar{s})$, there exist two equilibrium bid functions $b(\cdot|\bar{s}, X)$ and $\hat{b}(\cdot|\bar{s}, X)$. Then, by Theorem 1, both b and $\hat{b}$ satisfy (6)-(7). We apply now Lemma 1(ii) to $h(x) \equiv b(x|\bar{s}, X) - \hat{b}(x|\bar{s}, X)$. We have $h(H) = 0$. From (6) and (12), it is easy to see that $h(x) \leq 0 \Rightarrow h'(x) \leq 0$ for all $x \in [\max \bar{s}, H)$. Thus $h \geq 0$ on $[\max \bar{s}, H)$. But this logic applies also to $-h$, which implies $h \leq 0$ on $[\max \bar{s}, H)$. We therefore conclude that $b(x|\bar{s}, X) \equiv \hat{b}(x|\bar{s}, X)$ so that the second-stage equilibrium is necessarily unique. ■

3.3 The EPA symmetric equilibrium

We now consider the bidders' first stage strategies induced by the bid function b that has been derived in the preceding subsection. The first-stage strategy $(b_{(1)}, ..., b_{(n-2)})$ has been defined in (2). Our next theorem verifies that this strategy is well defined, and is an EPA equilibrium in the first stage given the second-stage equilibrium b.

Theorem 3 *The strategy $\bar{b}$ described in (1)-(2) is an EPA symmetric equilibrium.*

Proof. We prove the theorem in three steps.

Step 1. We first show that for all $\bar{s} \in [L, H]^{n-2}$ and $x \in [\max \bar{s}, H)$, $b(x|\bar{s}, p) - p$ strictly decreases in p as long as $b(x|\bar{s}, p) > p$.

To see this, suppose that $p < \hat{p}$ and that $b(r|\bar{s}, p) > p$ and $b(r|\bar{s}, \hat{p}) > \hat{p}$ for $r \in [\max \bar{s}, H)$ so that (6) holds with X replaced by p and $\hat{p}$ respectively. By $b_1 > 0$, the preceding two inequalities hold for all $x \in [r, H)$. We now apply Lemma 1(i) to $h(x) \equiv b(x|\bar{s}, p) - p - (b(x|\bar{s}, \hat{p}) - \hat{p})$. Because $h(H) = \hat{p} - p > 0$, there exists $\bar{x} < H$ such that $h > 0$ on $[\bar{x}, H]$. Now suppose $h(x) = 0$ for some $x \in [r, \bar{x})$. Then $b(x|\bar{s}, \hat{p}) - \hat{p} = b(x|\bar{s}, p) - p$ and therefore $b(x|\bar{s}, \hat{p}) > b(x|\bar{s}, p)$. We have seen from (12) that the function B defined in (8) satisfies $B_1 > 0$. Hence, $b_1(x|\bar{s}, \hat{p}) > b_1(x|\bar{s}, p)$ or $h'(x) < 0$. Since $h(\bar{x}) > 0$, Lemma 1(i) now implies $h(x) > 0$ on $[r, H)$. We conclude that $b(x|\bar{s}, p) - p$ is a strictly decreasing function of p for all $x \in [\max \bar{s}, H)$ as long as $b(x|\bar{s}, p) > p$.

Step 2. We next show that for $\hat{s}, \bar{s} \in [L, H]^{n-2}$ such that $\hat{s} \geq \bar{s}$, $b(x|\hat{s}, p) \geq b(x|\bar{s}, p)$ for all $x \in [\max \hat{s}, H]$ and p such that $b(x|\bar{s}, p) \geq p$.

Because $v(x, x, \hat{s}) \geq v(x, x, \bar{s})$, we have $B(b, x|\hat{s}, X) \leq B(b, x|\bar{s}, X)$ as can be seen in (8) and $b(H|\hat{s}, p) \geq b(H|\bar{s}, p)$ as can be seen in (7). Hence, by $B_1 > 0$, $b(x|\hat{s}, p) \leq b(x|\bar{s}, p)$ implies $b_1(x|\hat{s}, p) \leq b_1(x|\bar{s}, p)$. By Lemma 1(ii), we thus have $b(x|\hat{s}, p) \geq b(x|\bar{s}, p)$.

Step 3. The result of Step 2 and the fact that $b_1 > 0$ imply that $b_{(k)}(x; p)$ is a strictly increasing function of x. Moreover, $b_{(k)}(x; p)$ is continuous in x and p (see Remark 3). Hence, together with the result of Step 1 we have shown that the lower-signal bidders will drop out first, and that the strategy $(b_{(1)}, ..., b_{(n-2)})$ is well defined in (2). We now show that $\bar{b} = (b_{(1)}, ..., b_{(n-2)}, b)$ is an equilibrium.

Consider a bidder with signal x and assume that no bidder has dropped out yet. Then the question is: "If the bidder becomes a finalist at the current price p, does he expect a positive utility in the second stage?" Theorem 2 has ensured that all bidders will use the same bid function b in computing his expected utility. Now define $r(p)$ by $b_{(1)}(r(p); p) = p$ and $\bar{s}(p) = (r(p), ..., r(p))$. Clearly, as long as $b_{(1)}(x; p) > p$ so that $x > r(p)$, it is optimal for the bidder to stay because $U(x|\bar{s}(p), p) > U(r(p)|\bar{s}(p), p) = 0$. Once the price reaches the level where $b_{(1)}(x; p) = p$, however, then staying becomes a (weakly) dominated strategy because it leads to a higher bottom price than p in the second stage. It makes no difference in his expected utility if the bidder can quit later in the first stage. But if the bidder quits too late and becomes a finalist, with bottom price $X > p$, he must then bid as if his signal is higher than x in the second stage. This implies that his expected utility will be non-positive. (The best he might then do is to bid the bottom price X. But this is weakly dominated by quitting earlier: in case the other finalist also bids the bottom price, the random resolution of the tie could allocate the object to the bidder for too high a price, without any compensating premium.) This analysis applies to the situations where $k < n - 2$ bidders have dropped out, and where the subsequent strategy $b_{(k+1)}(x; p)$ determines the next one who will quit. We conclude that $\bar{b}$ is an EPA equilibrium. ■

Example: In order to gain some numerical insight, let us consider an example in which the bidders' utility exhibit constant absolute risk aversion (CARA):

$$u(x) = \frac{1 - \exp(-\lambda x)}{\lambda}, \qquad \lambda \in \mathbb{R}.$$

Suppose further that $v^i(s) = s_i$, and $\varphi(x) = \alpha x$ for some constant $\alpha \in (0, 1/2]$. The differential equation (6) then reduces to

$$b'_\lambda(x) = \frac{\exp(\lambda(b_\lambda(x) - x)) - 1}{\alpha\lambda} \frac{f(x)}{1 - F(x)} \tag{13}$$

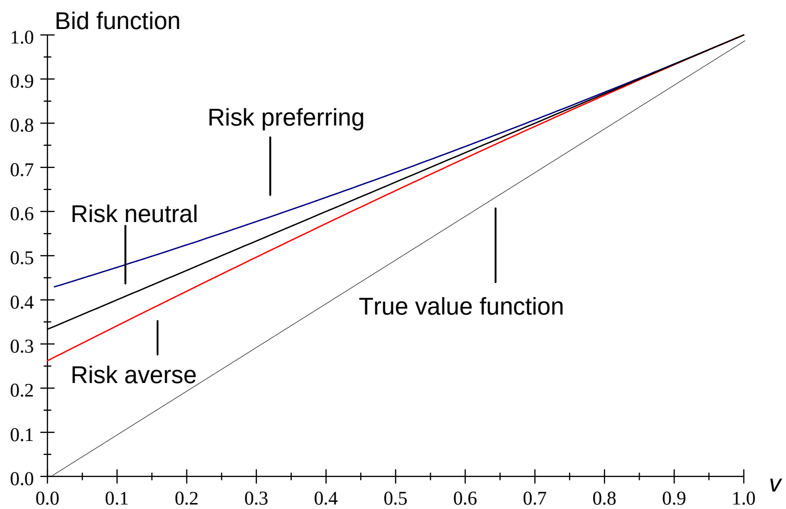

Figure 1: The premium induces the buyers to bid higher than their true values. The more risk tolerant the buyers are, the higher will be their bids. Here, F is assumed to be uniform on $[0, 1]$ and $\alpha = 0.5$. The risk averse bid function assumes $\lambda = 3$, and the risk seeking bid function assumes $\lambda = -3$.

where we observe that the bid function $b_\lambda(x)$ is independent of the revealed signals $s_{(1)}, ...s_{(n-2)}$ (because of private values), and independent of the bottom price X. This implies that for the case with private values and CARA utility, all bidders will adopt the strategy $(b_\lambda, ..., b_\lambda)$ in the first stage as well as the second stage.

The differential equation (13) can be solved explicitly to yield the EPA equilibrium b_λ. To see this, rearranging terms in (13), and multiplying both sides by $\exp(-\lambda b_\lambda(x))$, we obtain

$$\alpha\lambda b'_\lambda(x)\,(1 - F(x))\exp(-\lambda b_\lambda(x)) + f(x)\exp(-\lambda b_\lambda(x)) = f(x)\exp(-\lambda x)$$

Now multiply both sides of the above equation by $(1 - F(x))^{\frac{1}{\alpha}-1}$ to get

$$-\frac{\partial}{\partial x}\left(\alpha(1 - F(x))^{\frac{1}{\alpha}}\exp(-\lambda b_\lambda(x))\right) = (1 - F(x))^{\frac{1}{\alpha}-1} f(x)\exp(-\lambda x)$$

Then, integrating and rearranging terms yields the desired closed-form solution:

$$b_\lambda(v) = -\frac{1}{\lambda}\ln\left(\frac{1}{\alpha}\int_v^H \frac{e^{-\lambda y}}{1-F(y)}\left(\frac{1-F(y)}{1-F(v)}\right)^{\frac{1}{\alpha}} dF(y)\right)$$

Figure 1 depicts the bid functions of the risk averse, risk neutral, and risk preferring bidders. The figure confirms that the premium, in general, induces the bidders to bid higher than their true values. It also shows that the bids are uniformly higher (lower) if the bidders are more risk tolerant (averse).

4 The Premium Effects

We now compare the EPA with the standard EA in terms of their implied expected revenue and expected utilities of the participants. Since the losing bidders do not pay, the welfare effect of the premium is solely determined by the bidding strategies of the last two finalists. Suppose $n-2$ bidders have dropped out at a bottom price X for the EPA. These bidders, as well as their revealed signals $\bar{s} = (s_{(1)}, ...s_{(n-2)})$, will be the same in either an EA or an EPA. In the remainder of the paper, we therefore write $b(x)$ instead of $b(x|\bar{s}, X)$, and $v(x,y)$ instead of $v(x,y,\bar{s})$ to ease notation (unless needed for clarity).

The conditional expected utility at the screening level $r \geq s_{(n-2)}$ of each remaining bidder in the EA equals

$$\begin{aligned} W(x|r) &\equiv \frac{1}{1-F(r)}\int_r^x u(v(x,y)-v(y,y))dF(y) \\ &= \frac{1}{1-F(r)}\int_r^H u\left(\max(v(x,y)-v(y,y),0)\right)dF(y) \\ &= E_r\left[u(\max(v(x,y)-v(y,y),0))\right], \qquad (14) \end{aligned}$$

where the expectation $E_r(\cdot)$ is taken with respect to the conditional distribution of the opponent's signal y, i.e., $\left[F(y)-F(r)\right]/\left[1-F(r)\right]$.

Consider next an EPA with a premium function φ. In order to highlight the effects of the premium, let us define a "gamble" $\Phi(\cdot|x)$ conditional on one's signal

x. The payoff of Φ depends on the realization of $y \in [s_{(n-2)}, H]$ as follows:[15]

$$\Phi(y|x) = \begin{cases} v(y,y) - b(y) + \varphi(b(y) - X) & \text{if } y \in [s_{(n-2)}, x] \\ \varphi(b(x) - X) & \text{if } y \in (x, H] \end{cases}. \tag{15}$$

This function is discontinuous at $y = x$ for $x < H$, and it is a "jump" that occurs with a zero probability.

From (3), at any screening level $r \geq s_{(n-2)}$ the conditional expected utility of a finalist can now be written as

$$\begin{aligned}
&U(x|r, \bar{s}, X) \\
&= \frac{1}{1-F(r)} \int_r^x u\left(v(x,y) - v(y,y) + \Phi(y|x)\right) dF(y) \\
&+ \frac{1-F(x)}{1-F(r)} u\left(\varphi(b(x) - X)\right) \\
&= \frac{1}{1-F(r)} \int_r^H u\left(\max(v(x,y) - v(y,y), 0) + \Phi(y|x)\right) dF(y) \\
&= E_r\left[u(\max(v(x,y) - v(y,y), 0) + \Phi(y|x))\right].
\end{aligned} \tag{16}$$

Comparing the expected utility U in (16) to the expected utility W in (14) conditional on any screening level r, we find that the "gamble" Φ is entirely due to the premium offered, with $\Phi \equiv 0$ corresponding to the EA.

Our next theorem establishes a *net-premium* effect in the EPA. It predicts that as long as the second-stage EPA continues, the current screening level r reveals each finalist's conditional equilibrium expected utility for Φ—that is, when Φ is evaluated in *isolation* of other random payoffs. This result holds for arbitrary utility function u and premium function φ.

As can be seen from the proof of this theorem, the net-premium effect is essentially an "envelope theorem effect" as a consequence of incentive compatibility,

[15] If only the highest losing bidder receives the premium, then replacing the gamble Φ by Φ^0, defined as follows, will give the same predictions as in the subsequent theorems:

$$\Phi^0(y|x) = \begin{cases} v(y,y) - b(y) & \text{if } y \in [s_{(n-2)}, x] \\ \varphi(b(x) - X) & \text{if } y \in (x, H] \end{cases}$$

which reduces to a revenue equivalence result in the present context when the bidders are risk neutral. It should be stressed, however, that the net-premium effect is an *isolated* premium effect. Only in the special case of risk neutrality does the effect imply that the bidders will be indifferent about the premiums. In general, the premium will affect the expected utilities of the bidders in an EPA. This follows from the simple fact that for nonlinear utility functions, in general

$$
\begin{aligned}
&E_r\left[u(\max(v(x,y)-v(y,y),0)+\Phi(y|x))\right] \\
&\neq E_r\left[u(\max(v(x,y)-v(y,y),0))\right]+E_r\left[u(\Phi(y|x))\right]
\end{aligned}
$$

Theorem 4 (Net-premium effect) *The second-stage EPA equilibrium implies that for all $r \in [s_{(n-2)}, H]$, and $x \in [r, H]$,*

$$
E_r\left[u(\Phi(y|x))\right]=u(\varphi(b(r)-X)). \tag{17}
$$

Proof. In the second stage of the EPA, for any bottom price X and screening level $r \geq s_{(n-2)}$, the conditional expected utility of a finalist who has signal x and who bids as though his signal is z equals $\overline{U}(x,z|r,\bar{s},X)$ (see (4)). Differentiating $\overline{U}$ with respect to x gives

$$
\overline{U}_1(x,z|r,\bar{s},X)=\frac{1}{1-F(r)}\int_r^z u'\left(v(x,y)-b(y)+\varphi(b(y)-X)\right)v_1(x,y)dF(y).
$$

Because $\overline{U}$ is maximized at $z=x$ and $U(r|r,\bar{s},X)=u(\varphi(b(r)-X))$, incentive compatibility and the envelope theorem imply

$$
\begin{aligned}
&U(x|r,\bar{s},X) \\
&=U(r|r,\bar{s},X)+\int_r^x \overline{U}_1(z,z|r,\bar{s},X)dz \\
&=U(r|r,\bar{s},X)+\frac{1}{1-F(r)}\int_r^x\int_r^z u'\left(v(z,y)-b(y)+\varphi(b(y)-X)\right)v_1(z,y)dF(y)dz.
\end{aligned}
$$

Interchanging the order of integration we obtain

$$\begin{aligned}
&U(x|r,\bar{s},X) \\
&= U(r|r,\bar{s},X) + \frac{1}{1-F(r)}\int_r^x\int_y^x u'\left(v(z,y)-b(y)+\varphi(b(y)-X)\right)v_1(z,y)dzdF(y) \\
&= U(r|r,\bar{s},X) + \frac{1}{1-F(r)}\int_r^x u\left(v(x,y)-b(y)+\varphi(b(y)-X)\right)dF(y) \\
&- \frac{1}{1-F(r)}\int_r^x u\left(v(y,y)-b(y)+\varphi(b(y)-X)\right)dF(y). \qquad (18)
\end{aligned}$$

On the other hand, $U(x|r,\bar{s},X)$ has a direct expression given in (3). Thus, subtracting (3) from (18) yields

$$\begin{aligned}
E_r\left[u(\Phi(y|x))\right] &= \frac{1-F(x)}{1-F(r)}u\left(\varphi(b(x)-X)\right) \\
&+ \frac{1}{1-F(r)}\int_r^x u\left(v(y,y)-b(y)+\varphi(b(y)-X)\right)dF(y) \\
&= U(r|r,\bar{s},X) = u(\varphi(b(r)-X)).
\end{aligned}$$

■

The net-premium effect is useful for gaining insight into the competitive bidding behavior in the English premium auctions. Notice that the right-hand side of (17) equals the bidder's utility for the premium if he quits at the current screening level r. As long as the bidder has signal $x > r$, however, he will have no incentive to quit because staying in the auction gives him the same level of expected utility for the premium. In addition, from (16) we can see that there is an additional $\max(v(x,y)-v(y,y),0)$ to be possibly gained in case the opponent has a signal $y \in [r,x)$. In equilibrium, this reasoning is common knowledge, and it therefore offers both finalists the comfort to sit back and relax, watching their premium grow up with the price. It is also common knowledge that this bidding process will continue until one of the bidders no longer expects to win.

Another useful implication of the net-premium effect is that at the time when the first stage has just ended with a bottom price X, both finalists derive a conditional expected utility for the premium that must be equal to zero. This is because

$b(s_{(n-2)}) = X$ and thus $E_{s_{(n-2)}}[u(\Phi(y|x))] = 0$. A special case is where the bidders are risk neutral; then, the net-premium effect reduces to an equivalent statement of the revenue equivalence principle (e.g., Myerson (1981)) that $E_{s_{(n-2)}}[\Phi(y|x)] = 0$.

Just as the revenue equivalence principle offers a useful tool for comparing welfare implications of various auction policies, our net-premium effect theorem offers a handy tool for the subsequent analysis of the premium effects on the expected revenue and expected utilities of the seller and bidders in an EPA.

Proposition 1 *For arbitrary number n (> 2) of the bidders, and for arbitrary premium function φ, the expected revenue in any EPA equilibrium is lower (higher) when the bidders are more risk averse (preferring).*

Proof. Let $\hat{u}$ be another utility function satisfying the same assumptions as u, with an absolute risk aversion measure satisfying $-\hat{u}''/\hat{u}' > -u''/u'$ at all relevant income levels. Let $\hat{b}$ and $\hat{X}$ denote the bid function and the bottom price when the bidders' preferences are represented by $\hat{u}$, and define $\hat{\Phi}$ similar to Φ as in (15).

Let $r = s_{(n-2)}$. It is clear that in an EPA equilibrium, r is independent of the utility functional forms although the bottom price at which the first stage ends can be different as the utility function changes.

Denote by R and $\hat{R}$ the conditional expected payment of a finalist entering the second stage who has utility function u and $\hat{u}$, respectively. We have

$$\begin{aligned} R(x) &= \frac{1}{1-F(r)} \int_r^x [b(y) - \varphi(b(y) - X)]\, dF(y) - \frac{1-F(x)}{1-F(r)} \varphi(b(x) - X) \\ &= \frac{1}{1-F(r)} \left(\int_r^x [v(y,y) - \Phi(y|x)]\, dF(y) - (1-F(x))\, \varphi(b(x) - X) \right) \\ \hat{R}(x) &= \frac{1}{1-F(r)} \left(\int_r^x \left[v(y,y) - \hat{\Phi}(y|x)\right] dF(y) - (1-F(x))\, \varphi(\hat{b}(x) - \hat{X}) \right) \end{aligned}$$

Subtracting gives

$$\hat{R}(x) - R(x) = E_r\left(\Phi(y|x)\right) - E_r\left(\hat{\Phi}(y|x)\right).$$

We know from Theorem 4 that

$$E_r\left[\hat{u}(\hat{\Phi}(y|x))\right] = E_r\left[u(\Phi(y|x))\right] = 0. \tag{19}$$

Since $\hat{u}$ is more risk averse than u, the above equations imply

$$E_r\left(\Phi(y|x)\right) - E_r\left(\hat{\Phi}(y|x)\right) < 0 \tag{20}$$

and hence $\hat{R}(x) < R(x)$. Because the bidders are symmetric ex ante, this implies the conclusion of the theorem straightforwardly. ■

The reason why revenue decreases in the bidders' risk aversion can be seen from the expressions of R and $\hat{R}$: each finalist's conditional expected payment is the difference of the expected value of his opponent (in the event of winning) and the expected premium. Since the former has nothing to do with the utility functional forms, this difference is solely explained by the difference in the expected premiums. It follows then from the net-premium effect (19) that the expected premium in equilibrium increases in the bidders' risk aversion, resulting in a lower expected payment.

The following corollary is an immediate consequence of Proposition 1.

Corollary 1 *Given any number n (> 2) of the bidders, adding a premium φ to an EA increases the expected revenue when bidders are risk preferring, and decreases the expected revenue when bidders are risk averse.*

Proof. In an EA, for arbitrary utility function u, each bidder bids up to his true signal x. This strategy leads to the same expected revenue in an EA when bidders are risk neutral. By the revenue equivalence theorem, the expected revenue is also the same in an EPA where the bidders are risk neutral. Therefore, by Proposition 1, the expected revenue under any premium φ is higher (lower) than that without a premium when bidders are risk preferring (averse). ■

A straightforward implication of this corollary is that the problem of designing the "optimal premium function" that maximizes expected revenue, with the number of participants given, involves only a corner solution when the bidders are risk averse. In this case, the optimal premium function should be a constant zero.

The results obtained so far are quite general, as they hold without much restriction regarding the shape of the distribution, premium, and utility functions. For the subsequent results we will use some of the following three assumptions.

(A1) $\left(\frac{f(x)}{1-F(x)}\right)' > 0$ on (L, H).

For all $\bar{s} \in [L, H)^{n-2}$ and $x \in [\max \bar{s}, H)$,

(A2) $v''(x, x, \bar{s}) \leq 0$.[16]

(A3) $v_{12}(x, y, \bar{s}) \leq 0$, $\forall y \in [\max \bar{s}, x]$.

The assumption (A1) says that the hazard rate of F is increasing, and (A2) and (A3) assume that marginal values $v'(x, x, \bar{s})$ and $v_2(x, y)$ are nonincreasing in x.

We now examine the premium's effects on the riskiness of the participants' payoffs. Suppose a "gamble" $g(x)$ is added to an existing gamble $k(x)$, and in addition the two gambles move in opposite directions as the random variable x changes. Then, it is intuitive that all risk averse individuals would prefer the gamble $k + g$ to the gamble k if g is "favorable" (with nonnegative expected payoff), and all risk preferring individuals would prefer k to $k + g$ if g is "unfavorable" (with nonpositive expected payoff). These results seem to be reasonably well-known. However, as we could not find them anywhere in the literature, we present a lemma for completeness. The proof of the lemma can be found in the Appendix.[17]

Lemma 2 *For $-\infty < c < d < \infty$, let $k, g : [c, d] \to \mathbb{R}$ be continuous, with k increasing and g decreasing, or with k decreasing and g increasing. Let x be a random variable with continuous cumulative distribution $F : [c, d] \to [0, 1]$, and assume that*

$$\int_c^d g(x)dF(x) \geq (\leq)\, 0. \tag{21}$$

[16] It is defined here that $v''(x, x, \bar{s}) \equiv \frac{\partial^2}{\partial x^2} v(x, x, \bar{s})$. Recall also that $v'(x, x, \bar{s}) \equiv \frac{\partial}{\partial x} v(x, x, \bar{s})$.

[17] We only present the lemma in terms of weak inequalities. The results can be adapted straightforwardly to the cases with strict inequalities.

Then, for all (utility) functions $u : \mathbb{R} \to \mathbb{R}$ with $u' > 0$ and $u'' \leq (\geq)\, 0$,

$$\int_c^d u(k(x) + g(x))dF(x) \geq (\leq) \int_c^d u(k(x))dF(x).$$

Our next theorem presents a general result concerning the premium's effect on the riskiness of the payoffs to both the seller and the buyers. In light of Lemma 2, the result is termed the "SSD effect" because of its implication of second-order stochastic dominance (SSD).

Theorem 5 (SSD effect) *Suppose (A1)-(A2) hold. Then, in the second-stage EPA equilibrium, for any signal $x \in (s_{(n-2)}, H)$, the "premium gamble" $\Phi(y|x)$ is a strictly increasing function of the opponent's signal y on $(s_{(n-2)}, x)$.*

Proof. Fix $x \in (s_{(n-2)}, H)$. Differentiating Φ with respect to $y \in (s_{(n-2)}, x)$ and $y \in (x, H]$, respectively, gives

$$\Phi_1(y|x) = \begin{cases} v'(y,y) - [1 - \varphi'(b(y) - X)]b_1(y) & y \in (s_{(n-2)}, x) \\ 0 & y \in (x, H] \end{cases}.$$

We apply Lemma 1(i) to

$$h(y) = v'(y,y) - [1 - \varphi'(b(y) - X)]\, b_1(y).$$

Differentiating gives

$$h'(y) = v''(y,y) - (1 - \varphi')\, b_{11} + \varphi'' b_1^2.$$

Since $\varphi'' \leq 0$ and $v'' \leq 0$ (by (A2)), and $h(H) = v'(H,H)\left(1 - \frac{1-\varphi'(v(H,H,\bar{s})-X)}{1+\varphi'(v(H,H,\bar{s})-X)}\right) > 0$ (see (9)), it suffices to show that at any $y \in (s_{(n-2)}, x)$, $h(y) = 0$ implies $b_{11} > 0$ and hence $h'(y) < 0$.

Differentiating b_1 gives

$$\begin{aligned} b_{11}(y) = {} & B(b,y)\left(\frac{f(y)}{1 - F(y)}\right)' \\ & + B_1(b,y)b_1 - \frac{u'\left(v(y,y) - b + \varphi\right)}{u'\left(\varphi\right)\varphi'} v'(y,y) \end{aligned} \tag{22}$$

If $h(y) = 0$, then substituting $v'(y,y)/\left[1 - \varphi'(b(y) - X)\right]$ for $b_1(y)$ in (22), and using the inequality of (11), it is straightforward to verify that $B_1(b,y)b_1$ is no less than the last term in (22). Thus, by (A1),

$$b_{11}(y) \geq B(b,y)\left(\frac{f(y)}{1 - F(y)}\right)' > 0$$

Lemma 1(i) now implies that $h > 0$, and hence $\Phi_1(y|x) > 0$ for $y \in (s_{(n-2)}, x)$. ■

The conclusions of our next two propositions are straightforward consequences of the SSD effect and Lemma 2. The next proposition provides a clear case for the use of premium tactics in auctions, i.e., where the bidders are risk neutral and the seller is risk averse. The bidders will then be indifferent between EA and the EPA, while the seller benefits from the premium's role of reducing the riskiness of revenue.

Proposition 2 *Suppose (A1)-(A2) hold. Then, adding a premium φ to an English auction increases a risk averse seller's expected utility if the bidders are not "too" risk averse, and decreases a risk preferring seller's expected utility if the bidders are not "too" risk preferring.*

Proof. Let u_S denote the seller's utility function for income, and let $y = s_{(n-1)}$ and $r = s_{(n-2)}$ denote the second and third highest values from among the n bidders, respectively. In either an EA or an EPA, the same r is revealed as soon as there are two bidders remain. Hence, we focus on the seller's expected utilities conditional on r at the beginning of the second stage. The seller's utility is then uniquely determined by the realized value of y. Conditional on knowing $y \geq r$ in the second stage, the density function of y is $2\left[1 - F(y)\right] f(y)/\left[1 - F(r)\right]^2$. In what follows, $E(\cdot)$ denotes the expectation taken with respect to this density function.

In an EA when only two bidders remain, the seller's conditional expected utility is thus

$$E\left(u_S|\text{EA}\right) = \frac{2}{\left[1 - F(r)\right]^2}\int_r^H u_S\left(v(y,y)\right)\left[1 - F(y)\right]dF(y).$$

Likewise, in the second stage of an EPA the seller's conditional expected utility for the total net payment equals

$$\begin{aligned}
&E\left(u_S|\text{EPA}\right) \\
&= \frac{2}{[1-F(r)]^2} \times \int_r^H u_S\left(b(y) - 2\varphi(b(y) - X)\right)[1-F(y)]\,dF(y) \\
&= \frac{2}{[1-F(r)]^2} \int_r^H u_S\left(v(y,y) + \Psi(y)\right)[1-F(y)]\,dF(y),
\end{aligned}$$

where $\Psi(y) = b(y) - 2\varphi(b(y) - X) - v(y,y)$. We now apply Lemma 2 to $k(y) = v(y,y)$ and $g(y) = \Psi(y)$. Corollary 1 implies that the expected value of Ψ satisfies

$$E\left(\Psi(y)\right) \begin{cases} \geq 0 \text{ if } u'' \geq 0 \\ \leq 0 \text{ if } u'' \leq 0 \end{cases}.$$

By Theorem 5, g is decreasing because $y = s_{(n-1)} \in (r, s_{(n)}]$ and $\Psi'(y) = -\varphi' b_1(y) - \Phi_1(y|s_{(n)}) < 0$ for $y < s_{(n)}$. We also have k increasing. Thus, by Lemma 2,

$$E\left(u_S|\text{EPA}\right) \begin{cases} > E\left(u_S|\text{EA}\right) \text{ if } u'' \geq 0 \text{ and } u''_S < 0 \\ < E\left(u_S|\text{EA}\right) \text{ if } u'' \leq 0 \text{ and } u''_S > 0 \end{cases}. \tag{23}$$

Now, by a continuity argument, it can happen that when all players are risk averse, the seller prefers the EPA to the EA as long as the bidders are not "too" risk averse. This follows from the strict preference of the seller for the EPA when the bidders are risk neutral, as shown in (23). In this case, the premium's role of reducing the revenue risk is predominant for the seller. Conversely, when all players are risk preferring, it can happen that neither the seller nor the bidders would like to have the premium practice provided that the bidders are close to risk neutral. ■

When bidders are risk neutral, or when both EPA and EA have the same expected revenue, this proposition has an immediate corollary that the EPA entails a smaller variance of revenue than the EA. It follows from the fact that any mean-preserving spread (which is equivalent to being dominated in the sense of SSD) implies a larger variance (see Rothschild and Stiglitz, 1970, p. 241).[18] This generalizes a similar finding in Goeree and Offerman (2004) that the premium reduces the variance of payment, calculated under a uniform value distribution and linear premium rule.

Our last proposition concerns the bidders' preference for the auction forms.

Proposition 3 *Suppose (A1)-(A3) hold. Then, adding a premium φ to an EA increases the expected utility of the risk averse bidders, and decreases the expected utility of the risk preferring bidders.*

Proof. Since the bidders who drop out in the first stage of an EPA have a zero expected utility regardless of any premium, we focus on the two finalists' conditional expected utilities at the start of the second stage. Let $r = s_{(n-2)}$. By (A3), for $x, y \in [r, H)$ such that $x \geq y$ we have $v_2(x,y) \leq v_2(y,y)$. Hence, $v(x,y) - v(y,y)$ strictly decreases in y. By Theorem 5, $\Phi(y|x)$ strictly increases in $y \in [r,x)$. Proposition 1 implies that the expected value of Φ at the start of the second stage satisfies (see (20))

$$E_r\left(\Phi(y|x)\right) \begin{cases} > 0 \text{ if } u'' < 0 \\ < 0 \text{ if } u'' > 0 \end{cases}.$$

Comparing now the conditional expected utility U in (16) to the expected utility W in (14), and using Lemma 2, we conclude that

$$U(x|r) \begin{cases} > W(x|r) \text{ if } u'' < 0 \\ < W(x|r) \text{ if } u'' > 0 \end{cases}.$$

■

Intuitively, there are two effects of premium that are both favorable to the risk averse bidders: risk reduction and expected payment reduction. The opposite holds

[18] To see this in the present context, consider a seller's utility function of the form $u_S(w) = Kw + E(w - Ew)^2$ where K is some constant number. Then, for K sufficiently large, we have $u_S' > 0$ (since w is bounded) and $u_S'' > 0$. For risk neutral bidders, $E(w|\text{EPA}) = E(w|\text{EA}) \equiv Ew$. The second line of the inequalities in (23) then implies $E\left[(w - Ew)^2|\text{EPA}\right] < E\left[(w - Ew)^2|\text{EA}\right]$.

for risk preferring bidders. Back to the Example considered in Section 3, we see that the bid functions depicted in Figure 1 confirm the results of Proposition 3. Using the risk neutral bidders' bid function as the reference, we see that the premium induces the risk preferring bidders to bid "too high." Thus, the risk lovers will pay the seller a higher expected "net price" for the premium, resulting in lower expected utilities in comparison with the no-premium case (where they bid the true values in the private values setting). Likewise, the risk averse bidders bid "too low," and hence the seller's expected revenue is lower, and the bidders are uniformly better off with, rather than without, a premium.

5 Concluding Discussion

This paper presents an English premium auction model in a symmetric interdependent values setting, allowing both the seller and the bidders to exhibit more general risk preferences than risk neutrality. The existence and uniqueness of the symmetric equilibrium for the EPA is established, along with some in-depth analyses of the effects of premium in relation to the bidders' risk preferences. The net premium effect and the SSD effect that emerge from this study lead to a number of clear-cut results.

We conclude from this study that a seller facing ex ante symmetric bidders may consider a premium auction in a situation where he is risk averse and where he has some good reason to believe that the bidders are risk neutral, risk preferring, or not too risk averse. As long as the benefit of risk reduction outweighs the potential cost of a lower expected revenue, the premium auction will be preferred by both the seller and the buyers. A risk neutral seller facing risk averse bidders is better advised to use a standard auction, though.

The model presented in this paper has assumed a fixed number of bidders. A natural extension is to endogenize the entry decision of the potential bidders when they face certain costs to participate in the auction, e.g., the cost of learning their signals (e.g., McAfee and McMillan, 1987; Levin and Smith, 1994; Milgrom, 2004). If in such a model the bidders are risk averse, the seller may employ a premium auction to encourage entry and to enhance the bidding competition. The present study will then serve as a first step toward such extensions.

Appendix

Proof of Theorem 1. Fix $\bar{s} \in [L, H]^{n-1}$ and X. For notational convenience, we write $b(x)$ instead of $b(x|\bar{s}, X)$, and $v(x, y)$ instead of $v(x, y, \bar{s})$ for short. The proof involves four steps.

Step 1. We first show that if $b(x)$ is an equilibrium, then it is continuous on $[\max \bar{s}, H)$.[19]

Fix any screening level $r \in [\max \bar{s}, H)$. By (5) we must have

$$\overline{U}(x, x|r, \bar{s}, X) - \overline{U}(x, z|r, \bar{s}, X) \geq 0, \quad \forall x, z \in [r, H),$$

which is equivalent to (see (4))

$$\begin{aligned} & [1 - F(x)] \left[u\left(\varphi(b(z) - X)\right) - u\left(\varphi(b(x) - X)\right) \right] \\ & \leq \int_z^x u\left(v(x, y) - b(y) + \varphi(b(y) - X)\right) dF(y) \\ & + [F(z) - F(x)]\, u\left(\varphi(b(z) - X)\right) \end{aligned} \tag{24}$$

Likewise, (5) implies $\overline{U}(z, z|r, \bar{s}, X) - \overline{U}(z, x|r, \bar{s}, X) \geq 0$, which is equivalent to

$$\begin{aligned} & [1 - F(z)] \left[u\left(\varphi(b(x) - X)\right) - u\left(\varphi(b(z) - X)\right) \right] \\ & \leq \int_x^z u\left(v(x, y) - b(y) + \varphi(b(y) - X)\right) dF(y) \\ & + [F(x) - F(z)]\, u\left(\varphi(b(x) - X)\right) \end{aligned} \tag{25}$$

[19] The approach taken here is different than Maskin and Riley (1984, p. 1485-1486) for the case of private values.

Because v and b are bounded, taking limit as $z \to x$ the right-hand sides of (24) and (25) vanish. And because u and φ are continuous and increasing, these imply

$$0 \leq u\left(\varphi(\lim_{z\to x} b(z) - X)\right) - u\left(\varphi(b(x) - X)\right) \leq 0$$

and therefore $\lim_{z\to x} b(z) = b(x)$ for all $x \in [\max \bar{s}, H)$.

Step 2. We next show that b is differentiable, and necessarily satisfies (6).

Because u and φ are differentiable, by the mean value theorem we have

$$u\left(\varphi(b(z) - X)\right) - u\left(\varphi(b(x) - X)\right) = u'\left(\varphi(\xi)\right)\varphi'(\xi)\left[b(z) - b(x)\right] \tag{26}$$

where $\xi \to b(x) - X$ as $z \to x$ because $b(\cdot)$ is continuous. Now suppose $z > x$ ($z < x$). Then substituting (26) into (24) and (25) we obtain

$$\frac{1}{1 - F(z)}\Gamma(x, z) \leq (\geq) \frac{b(z) - b(x)}{z - x} \leq (\geq) \frac{1}{1 - F(x)}\Gamma(x, z) \tag{27}$$

where

$$\Gamma(x, z) \equiv \frac{1}{u'\left(\varphi(\xi)\right)\varphi'(\xi)} \times \\ \left(\frac{\int_z^x u\left(v(x, y) - b(y) + \varphi(b(y) - X)\right) dF(y)}{z - x} + \frac{F(z) - F(x)}{z - x} u\left(\varphi(b(z) - X)\right)\right)$$

As $z \to x$, both sides of (27) have the same limit, which is equal to the right-hand side of (6). Hence, $b(\cdot)$ is differentiable and necessarily satisfies (6).

Step 3. We next show that $b_1 > 0$.

From (6) and (8) it is easy to see that $b_1(x) > 0$ is equivalent to $b(x) > v(x, x)$.

We now apply Lemma 1(i) to $h(x) \equiv b(x) - v(x, x)$. Since $h(H) = 0$ and $h'(H) = b_1(H) - v'(H, H, \bar{s}) < 0$ (see (9)), we have $h > 0$ on $[\bar{x}, H)$ for some $\bar{x} < H$. If $h(x) = 0$ for some $x < \bar{x}$, then (6) and (8) imply $b_1(x) = 0 < v'(x, x)$ and hence $h'(x) < 0$. Lemma 1(i) now implies $h(x) > 0$ and therefore $b(x) > v(x, x)$, or equivalently, $b_1(x) > 0$ on $[\max \bar{s}, H)$.

Step 4. Finally, we show sufficiency and the boundary condition.

Differentiating $\overline{U}(x, z|r, \bar{s}, X)$ (see (4)) with respect to z gives

$$\begin{aligned}
&\overline{U}_2(x, z|r, \bar{s}, X) \\
&= \frac{f(z)}{1-F(r)} \left[u\left(v(x,z) - b(z) + \varphi(b(z) - X)\right) - u\left(\varphi(b(z) - X)\right)\right] \\
&+ \frac{1-F(z)}{1-F(r)} u'\left(\varphi(b(z) - X)\right) \varphi'(b(z) - X) b_1(z). \qquad (28)
\end{aligned}$$

Because the right-hand side of the above expression strictly increases in x,

$$\overline{U}_2(x, x|r, \bar{s}, X) = 0 \text{ implies } \overline{U}_2(x, z|r, \bar{s}, X) \begin{cases} > 0 \text{ if } z < x \\ = 0 \text{ if } z = x \\ < 0 \text{ if } z > x \end{cases}.$$

As $z \uparrow H$, the term in (28) vanishes (because $\overline{U}_2(x, x|r, \bar{s}, X) = 0$ implies $\lim_{x \uparrow H} b_1(x) < \infty$, as shown in (9)). We then have

$$\begin{aligned}
&\overline{U}_2(H, H|r, \bar{s}, X) = \frac{f(H)}{1-F(r)} \times \\
&\left[u\left(v(H,H) - b(H) + \varphi(b(H) - X)\right) - u\left(\varphi(b(H) - X)\right)\right] \\
&\lesseqgtr 0 \text{ for } b(H) \gtreqless H.
\end{aligned}$$

We conclude that b is a second-stage equilibrium if and only if it satisfies (6)-(7). ■

Proof of Lemma 2. Let $a = \min\left\{k(c), \min_{x \in [c,d]} (k(x) + g(x))\right\}$ and $b = \max\left\{k(c), \max_{x \in [c,d]} (k(x) + g(x))\right\}$. Define P and Q to be the induced cumulative distributions of random variables $k(x)$ and $k(x) + g(x)$, respectively. Then, for $z \in [a, b]$,

$$\begin{aligned}
P(z) &= \Pr(k(x) \leq z) \\
Q(z) &= \Pr(k(x) + g(x) \leq z)
\end{aligned}$$

These allow us to express the expected values of k and $k + g$ as

$$\begin{aligned}
\int_c^d k(x) dF(x) &= \int_a^b z dP(z) \\
\int_c^d \left[k(x) + g(x)\right] dF(x) &= \int_a^b z dQ(z)
\end{aligned}$$

We focus on the case with u concave and g favorable, and with k increasing and g decreasing. The rest of the cases can be proved analogously.

It suffices to show that Q dominates P in terms of second-order stochastic dominance (e.g., Hadar and Russell, 1969; Rothschild and Stiglitz, 1970). That is,

$$\int_a^y [P(z) - Q(z)]\, dz \geq 0 \quad \forall y \in [a, b] \tag{29}$$

Because g satisfies (21) and is decreasing, there must be an $x_0 \in (c, d]$ such that $g(x) \geq 0$ for $x < x_0$ and $g(x) \leq 0$ for $x > x_0$. Because k is increasing, for $z \geq k(x_0)$

$$k(x) > z \Rightarrow k(x) > k(x_0) \Rightarrow x > x_0 \Rightarrow g(x) \leq 0,$$

Consequently, $k(x) + g(x) > z$ implies $k(x) > z$. This is equivalent to $1 - Q(z) \leq 1 - P(z)$ or $Q(z) \geq P(z)$ for $z \in [k(x_0), b]$.

By assumption (21), we have

$$\int_a^b [P(z) - Q(z)]dz = \int_a^b zd\left(Q(z) - P(z)\right) = \int_c^d g(x)dF(x) \geq 0$$

Thus, by Lemma 1(ii), the inequality in (29) holds for all $y \in [k(x_0), b]$ because

$$\int_a^y [P(z) - Q(z)]dz \leq 0 \Rightarrow \frac{d}{dy}\int_a^y [P(z) - Q(z)]dz \leq 0$$

A similar analysis shows that for $z < k(x_0)$, $k(x) + g(x) \leq z$ implies $k(x) \leq z$, which is equivalent to $Q(z) \leq P(z)$ for $z \in [a, k(x_0)]$. The stochastic dominance condition in (29) is thus established. We conclude (e.g., by Hadar and Russell, 1969) that for all utility functions u such that $u' > 0$ and $u'' \leq 0$,

$$\int_c^d u(k(x) + g(x))dF(x) = \int_a^b u(z)dQ(z) \geq \int_a^b u(z)dP(z) = \int_c^d u(k(x))dF(x)$$

■

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